Solution to Assignment 2, MMAT5520

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Exercise 2.1:

1. Find the reduced row echelon form of the following matrices.

(f) $\begin{pmatrix} 1 & -2 & -4 & 5 \\ -2 & 4 & -3 & 1 \\ 3 & -6 & -1 & 4 \end{pmatrix}$ (g) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 & 4 \\ -1 & -2 & -1 & -2 & -3 \end{pmatrix}$

Soution:

(f)

$$\begin{pmatrix} 1 & -2 & -4 & 5 \\ -2 & 4 & -3 & 1 \\ 3 & -6 & -1 & 4 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 2R_1} \begin{pmatrix} 1 & -2 & -4 & 5 \\ 0 & 0 & -11 & 11 \\ 0 & 0 & 11 & -11 \end{pmatrix} \xrightarrow{R_2 \to -\frac{1}{11}R_2} \begin{pmatrix} 1 & -2 & -4 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 11 & -11 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 11R_2} \begin{pmatrix} 1 & -2 & -4 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 4R_2} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(g)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 & 4 \\ -1 & -2 & -1 & -2 & -3 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.Solve the following systems of linear equations.

(c) $\begin{cases} 2x_1 - x_2 + 5x_3 = 15 \\ x_1 + 3x_2 - x_3 = 4 \\ x_1 - 4x_2 + 6x_3 = 11 \\ 3x_1 + 9x_2 - 3x_3 = 12 \end{cases}$ (e) $\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$

Soution:

(c)

$$\left(\begin{array}{ccc|c} 2 & -1 & 5 & 15 \\ 1 & 3 & -1 & 4 \\ 1 & -4 & 6 & 11 \\ 3 & 9 & -3 & 12 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 2 & -1 & 5 & 15 \\ 1 & -4 & 6 & 11 \\ 3 & 9 & -3 & 12 \end{array}\right) \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -7 & 7 & 7 \\ 0 & -7 & 7 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \to -\frac{1}{7}R_2} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & -7 & 7 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - 3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Now, x_1, x_2 are leading variables while x_3 is a free variable, the solution of the system is

$$(x_1, x_2, x_3) = (7 - 2\alpha, \alpha - 1, \alpha), \quad \alpha \in \mathbb{R}.$$

(e)

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{2} R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 4R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Now, x_1, x_4 are leading variables while x_2, x_3 are free variables, the solution of the system is

$$(x_1, x_2, x_3, x_4) = (2\alpha - \beta, \alpha, \beta, 1), \quad \alpha, \beta \in \mathbb{R}.$$

Exercise 2.2:
3. Let A be a square matrix. Prove that A can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

Proof. Let

$$B = \frac{1}{2}(A + A^T), \quad C = \frac{1}{2}(A - A^T),$$

then we have

$$A = B + C,$$

and

$$B^T = B, \quad C^T = -C.$$

Exercise 2.3:
1.Find the inverse of the following matrices.

(b) $\begin{pmatrix} 5 & 7 \\ 4 & 6 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & -3 & -3 \\ -1 & 1 & 2 \\ 2 & -3 & -3 \end{pmatrix}$

Soution:
(b)

$$\left(\begin{array}{cc|cc} 5 & 7 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 4 & 6 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 4R_1} \left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & 2 & -4 & 5 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & \frac{5}{2} \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & 3 & -\frac{7}{2} \\ 0 & 1 & -2 & \frac{5}{2} \end{array}\right).$$

Therefore

$$A^{-1} = \begin{pmatrix} 3 & -\frac{7}{2} \\ -2 & \frac{5}{2} \end{pmatrix}.$$

(e)

$$\left(\begin{array}{ccc|ccc} 1 & -3 & -3 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 2 & -3 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -2 & -1 & 1 & 1 & 0 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2+R_3}\left(\begin{array}{ccc|ccc} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 3 & 3 & -2 & 0 & 1 \end{array}\right)\xrightarrow{R_3\to R_3-3R_2}\left(\begin{array}{ccc|ccc} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & -3 & 1 & -3 & -2 \end{array}\right)$$

$$\xrightarrow{R_3\to -\frac{1}{3}R_3}\left(\begin{array}{ccc|ccc} 1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & 1 & \frac{2}{3} \end{array}\right)\xrightarrow[R_1\to R_1+3R_3]{R_2\to R_2-2R_3}\left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -\frac{1}{3} & -1 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 1 & \frac{2}{3} \end{array}\right)$$

$$\xrightarrow{R_1\to R_1+3R_2}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{3} & -1 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 1 & \frac{2}{3} \end{array}\right).$$

Therefore

$$A^{-1}=\begin{pmatrix} -1 & 0 & 1 \\ -\frac{1}{3} & -1 & -\frac{1}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \end{pmatrix}.$$

2. Solve the following systems of linear equations by finding the inverse of the coefficient matrices.

(b) $\begin{cases} 5x_1 & + & 3x_2 & + & 2x_3 & = & 4 \\ 3x_1 & + & 3x_2 & + & 2x_3 & = & 2 \\ & & x_2 & + & x_3 & = & 5 \end{cases}$

Soution:

Let $A=\begin{pmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}$, $x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b=\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$, then $Ax=b$.

$$\left(\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{ccc|ccc} 3 & 3 & 2 & 0 & 1 & 0 \\ 5 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)\xrightarrow[R_2\leftrightarrow R_3]{R_1\to \frac{1}{3}R_1}\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 5 & 3 & 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3-5R_1}\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & -\frac{4}{3} & 1 & -\frac{5}{3} & 0 \end{array}\right)\xrightarrow{R_3\to R_3+2R_2}\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & \frac{2}{3} & 1 & -\frac{5}{3} & 2 \end{array}\right)$$

$$\xrightarrow{R_3\to \frac{3}{2}R_3}\left(\begin{array}{ccc|ccc} 1 & 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right)\xrightarrow[R_1\to R_1-\frac{2}{3}R_3]{R_2\to R_2-R_3}\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right)$$

$$\xrightarrow{R_1\to R_1-R_2}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & -2 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{5}{2} & 3 \end{array}\right).$$

Therefore $A^{-1}=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{pmatrix}$, and hence

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=A^{-1}b=\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & -2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}=\begin{pmatrix} 1 \\ -11 \\ 16 \end{pmatrix}$$

Exercise 2.4:

1. Evaluate the following determinants.

(c) $\begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix}$

Soution:

$$\begin{vmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{vmatrix} \xlongequal{C_4 \to C_4 - 2C_2} \begin{vmatrix} 5 & 3 & 0 & 0 \\ 4 & 6 & 4 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \end{vmatrix} = 0.$$

3. For the given matrix A, evaluate A^{-1} by finding the adjoint matrix $adjA$ of A.

(b) $A = \begin{pmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$

Soution:

$$\det A = 2 \times \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2 \times (2 - 0) = 4,$$

and

$$adjA = \begin{pmatrix} \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & 5 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} -3 & 5 \\ 1 & -3 \end{vmatrix} \\ -\begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix}.$$

Therefore

$$A^{-1} = \frac{adjA}{\det A} = \frac{1}{4} \begin{pmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

4. Use Cramer's Rule to solve the following linear systems.

(a) $\begin{cases} 4x_1 - x_2 - x_3 = 1 \\ 2x_1 + 2x_2 + 3x_3 = 10 \\ 5x_1 - 2x_2 - 2x_3 = -1 \end{cases}$

Soution:

$$\det A = \begin{vmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 5 & -2 & -2 \end{vmatrix} \xlongequal{C_2 \to C_2 - C_3} \begin{vmatrix} 4 & 0 & -1 \\ 2 & -1 & 3 \\ 5 & 0 & -2 \end{vmatrix} = -\begin{vmatrix} 4 & -1 \\ 5 & -2 \end{vmatrix} = 3.$$

Thus by Cramer's Rule,

$$x_1 = \frac{1}{3} \begin{vmatrix} 1 & -1 & -1 \\ 10 & 2 & 3 \\ -1 & -2 & -2 \end{vmatrix} \xlongequal{C_2 \to C_2 - C_3} \frac{1}{3} \begin{vmatrix} 1 & 0 & -1 \\ 10 & -1 & 3 \\ -1 & 0 & -2 \end{vmatrix} = -\frac{1}{3} \begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} = 1,$$

$$x_2 = \frac{1}{3} \begin{vmatrix} 4 & 1 & -1 \\ 2 & 10 & 3 \\ 5 & -1 & -2 \end{vmatrix} \xlongequal[C_3 \to C_3 + C_2]{C_1 \to C_1 - 4C_2} \frac{1}{3} \begin{vmatrix} 0 & 1 & 0 \\ -38 & 10 & 13 \\ 9 & -1 & -3 \end{vmatrix} = -\frac{1}{3} \begin{vmatrix} -38 & 13 \\ 9 & -3 \end{vmatrix} = 1,$$

$$x_3 = \frac{1}{3} \begin{vmatrix} 4 & -1 & 1 \\ 2 & 2 & 10 \\ 5 & -2 & -1 \end{vmatrix} \xlongequal[C_2 \to C_2 + C_3]{C_1 \to C_1 - 4C_3} \frac{1}{3} \begin{vmatrix} 0 & 0 & 1 \\ -38 & 12 & 10 \\ 9 & -3 & -1 \end{vmatrix} = \frac{1}{3} \begin{vmatrix} -38 & 12 \\ 9 & -3 \end{vmatrix} = 2.$$

Exercise 2.5:

1. Find the equation of the parabola of the form $y = ax^2 + bx + c$ passing through the given set of three points.

(a) $(0, -5), (2, -1), (3, 4)$.

Soution: The required equation is

$$\begin{vmatrix} 1 & x & x^2 & y \\ 1 & 0 & 0 & -5 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 4 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & x & x^2 & y \\ 1 & 0 & 0 & -5 \\ 0 & 2 & 4 & 4 \\ 0 & 3 & 9 & 9 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & x & x^2 & y \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 3 & 3 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & x & x^2 & y \\ 1 & 0 & 0 & -5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 0,$$

$$y = x^2 - 5.$$

2. Find the equation of the circle passing through the given set of three points.

(a) $(-1, -1), (6, 6), (7, 5)$

Soution: The equation of required circle is

$$\begin{vmatrix} 1 & x & y & x^2 + y^2 \\ 1 & -1 & -1 & (-1)^2 + (-1)^2 \\ 1 & 6 & 6 & 6^2 + 6^2 \\ 1 & 7 & 5 & 7^2 + 5^2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & x & y & x^2 + y^2 \\ 1 & -1 & -1 & 2 \\ 1 & 6 & 6 & 72 \\ 1 & 7 & 5 & 74 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & x & y & x^2 + y^2 \\ 1 & -1 & -1 & 2 \\ 0 & 7 & 7 & 70 \\ 0 & 8 & 6 & 72 \end{vmatrix} = 0,$$

$$\cdots$$

$$\begin{vmatrix} 1 & x & y & x^2 + y^2 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{vmatrix} = 0,$$

$$x^2 + y^2 - 6x - 4y - 12 = 0.$$